ZACHARY SPACES $\mathcal{Z}^p[\mathbb{R}^\infty]$ AND SEPARABLE BANACH SPACES

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Abstract. We construct Zachary space in $\mathbb{R}^\infty$ and find that this is a Banach space of functions of bounded mean oscillation with order $p, 1 \leq p \leq \infty$ containing the function of bounded mean oscillation $BMO[\mathbb{R}_I^\infty]$ as a dense continous embedding.As an application of $\mathbb{R}_I^\infty$ we construction $\mathcal{B}$ where $\mathcal{B}$ is separable Banach space and finally we construct $\mathcal{Z}^p[\mathcal{B}]$.

Keywords and phrases: Dense; Continuous embedding; Banach space

1. Introduction and Preliminaries

In [YY] Y.Yamasaki give a theory of $\mathbb{R}^\infty$ is very complicated to workout in the problem. In 2008, Gill and Zachary [GZ] introduced a new theory of lebesgue measure on $\mathbb{R}^\infty$; the construction of which is nearly same as the development of lebesgue measure on $\mathbb{R}^n$. This theory can be useful in formulating a new class of spaces which will provide a Banach Spaces which will provide a Banach Space structure for Henstock-Kurzweil(HK) integrable functions. In [GZ] again they define Zachary spaces in $\mathbb{R}^n$, they find that Zachary space contained the BMO space as a dense continuous embedding. In [GM] define a new approach of $\mathbb{R}_I^n$ completely parallal to $\mathbb{R}^n$, with the help of [GM] we construct alternate approach of $\mathbb{R}^\infty$.

In [GRA], they mention that the space BMO is not natural Banach space and not

separable although BMO is similar type of space of L^p. In [DC] find duality space of BMO space is Hardy space (H^1).

Definition 1.1. [GZ] *We define $\mathbb{R}^n_I = \mathbb{R}^n \times I_n$. If T is a linear transformation on $\mathbb{R}^n$ and $A_n = A \times I_n$, we define T_I on $\mathbb{R}^n_I$ by $T_I[A_n] = T[A]$, we define $\mathcal{B}[\mathbb{R}^n_I]$ to be the Borel $\sigma-$algebra for $\mathbb{R}^n_I$, where the topology for $\mathbb{R}^n_I$ is defined via the following class of open sets $D_n = \{U \times I_n : U$ is open in $\mathbb{R}^n\}$. For any $A \in \mathcal{B}[\mathbb{R}^n], \}$we define $\lambda_\infty(A_n)$ on $\mathbb{R}^n_I$ by product measure $\lambda_\infty(A_n) = \lambda_n(A) \times \Pi_{i=n+1}^{\infty} \lambda_I(I) = \lambda_n(A)$*

Theorem 1.2. [GM] *$\lambda_\infty(.)$ is a measure on $\mathcal{B}[\mathbb{R}^n_I]$ is equivalent to n-dimensional Lebesgue measure on $\mathbb{R}^n$.*

Corollary 1.3. *The measure $\lambda_\infty(.)$ is both translationally and rotationally invariant on $(\mathbb{R}^n_I, \mathcal{B}[\mathbb{R}^n_I])$ for each $n \in \mathbb{N}$.*

Thus we can construct a theory on $\mathbb{R}^n_I$ that completely paralleis that on $\mathbb{R}^n$.

Since $\mathbb{R}^n_I \subset \mathbb{R}^{n+1}_I$, we have an increasing sequence so we define $\widehat{\mathbb{R}}^\infty_I = \lim_{n\to\infty} \mathbb{R}^n_I = \bigcup_{k=1}^{\infty} \mathbb{R}^k_I$.

In [GM] , it is shown that we can extend the measure $\lambda_\infty(.)$ to $\mathbb{R}^\infty$.

We take measurable functions as follows

Let $x = (x_1, x_2, ...) \in \mathbb{R}^\infty_I$. let $I_n = \Pi_{k=n+1}^{\infty}[\frac{-1}{2}, \frac{1}{2}]$ and let $h_n(\widehat{x}) = \chi_{I_n}(\widehat{x})$ where $\widehat{x} = (x_i)_{i=n+1}^{\infty}$.

Definition 1.4. [GM]*Let M^n represents the class of Lebesgue measurable functions on $\mathbb{R}^n$. If $x \in \mathbb{R}^\infty_I$ and $f^n \in M^n$, let $\widehat{x} = (x_i)_{i=1}^n$ measurable function of order n(or e_n-tame) on $\mathbb{R}^\infty_I$ by $f(x) = f^n(\overline{x}) \otimes h_n(\widehat{x})$. we let*

$$M^n_I = \{f(x) \ : f(x) = f^n(\overline{x}) \otimes h_n(\widehat{x}) \ , x \in \mathbb{R}^\infty_I\}$$

be the class of all e_n-tame functions.

Definition 1.5. *A function $f : \mathbb{R}_I^\infty \to \mathbb{R}$ is said to be measurable and we write $f \in M_I$, if there is a sequence $\{f_n \in M_I^n\}$ of e_n-tame functions, such that $\lim_{n\to\infty} f_n(x) \to f(x)$ λ_∞-a.e.*

As an application of $\mathbb{R}_I^\infty$, we can construct $\mathcal{B}_J^n$ where $\mathcal{B}$ is separable Banach space. Let $\{e_k\}$ be an S-basis for $\mathcal{B}$, and let $x = \sum_{n=1}^\infty x_n e_n$, and from $\mathcal{P}_n(x) = \sum_{k=1}^n x_k e_k$ and $\mathcal{Q}_n x = (x_1, x_2, .., x_n)$ then $\mathcal{B}_J^n$ defined

$$\mathcal{B}_J^n = \{\mathcal{Q}_n x : x \in \mathcal{B}\} \times J^n$$

with norm

$$||(x_k)||_{\mathcal{B}_J^n} = \max_{1\le k\le n} ||\sum_{i=1}^k x_i e_i||_{\mathcal{B}} = \max_{1\le k\le n} ||\mathcal{P}(x)||_{\mathcal{B}}$$

As $\mathcal{B}_J^n \subset \mathcal{B}_J^{n+1}$ so we can set $\mathcal{B}_J^\infty = \bigcup_{n=1}^\infty \mathcal{B}_J^n$ and $\mathcal{B}_J$ is a subset of $\mathcal{B}_J^\infty$.

We set $\mathcal{B}_J$ as $\mathcal{B}_J = \{(x_1, x_2, ..) : \sum_{k=1}^\infty x_k e_k \in \mathcal{B}\}$ and norm on $\mathcal{B}_J$ by

$$||x||_{\mathcal{B}_J} = \sup_n ||\mathcal{P}_n(x)||_{\mathcal{B}} = ||x||_{\mathcal{B}}$$

If we consider $\mathcal{B}[\mathcal{B}_J^\infty]$ be the smallest σ-algebra containing $\mathcal{B}_J^\infty$ and define $\mathcal{B}[\mathcal{B}_J] = \mathcal{B}[\mathcal{B}_J^\infty] \cap \mathcal{B}_J$ then by a known result

$$||x||_{\mathcal{B}} = \sup_n ||\sum_{k=1}^n x_k e_k||_{\mathcal{B}} \tag{1}$$

is an equivalent norm on $\mathcal{B}$.

Proposition 1.6. *When $\mathcal{B}$ carries the equivalent norm (1) , the operator*

$$T : (\mathcal{B}, ||.||_{\mathcal{B}}) \to (\mathcal{B}_J, ||.||_{\mathcal{B}_J}$$

defined by $T(x) = (x_k)$ *is an isometric isomorphism from* $\mathcal{B}$ *onto* $\mathcal{B}_J$.

This shows that every Banach space with an S-basis has a natural embedding in $\mathbb{R}_I^\infty$

So we call $\mathcal{B}_J$ *the canonical representation of* $\mathcal{B}$ *in* $\mathbb{R}_I^\infty$ *With* $\mathcal{B}[\mathcal{B}_J] = \mathcal{B}_J \cap \mathcal{B}[\mathcal{B}_J^\infty]$ *we define* $\sigma-$*algebra generated by* $\mathcal{B}$ *and associated with* $\mathcal{B}[\mathcal{B}_J]$ *by*

$$\mathcal{B}_J[\mathcal{B}] = \{T^{-1}(A) \ | A \in \mathcal{B}[\mathcal{B}_J]\} = T^{-1}\{\mathcal{B}[\mathcal{B}_J]\}$$

since $\lambda_\infty(A_J^n) = 0$ *for* $A_J^n \in \mathcal{B}[\mathcal{B}_J^n]$ *with* A_J^n *compact , we see* $\lambda_\infty(\mathcal{B}_J^n) = 0,\ n \in \mathbb{N}$. *So,* $\lambda_\infty(\mathcal{B}_J) = 0$ *for every Banach space with an S-basis. Thus the restriction of* λ_∞ *to* $\mathcal{B}_J$ *will not induce a non trivial measure on* $\mathcal{B}$

The definition 1.5 *highlights our requirement that all functions on infinite dimensional space must be constructively defined as (essentially) finite dimensional limits.*

Using this definition $\mathcal{Z}^p[\mathbb{R}_I^n]$ we see that $\mathcal{Z}^p[\mathbb{R}_I^n] \subset \mathcal{Z}^p[\mathbb{R}_I^{n+1}]$

Thus we can define

$$\mathcal{Z}^p[\widehat{\mathbb{R}}_I^\infty] = \bigcup_{n=1}^{\infty} \mathcal{Z}^p[\mathbb{R}_I^n]$$

Definition 1.7. *We say that , for* $1 \leq p \leq \infty$. *a measurable function* $f \in \mathcal{Z}^p[\mathbb{R}_I^\infty]$ *if there is a Cauchy-sequence* $\{f_n\} \subset \mathcal{Z}^p[\widehat{\mathbb{R}}_I^\infty]$ *with* $f_n \in \mathcal{Z}^p[\mathbb{R}_I^n]$ *and* $\lim_{n\to\infty} f_n(x) = f(x)$ *.*λ_∞*-a.e.*

Definition 1.8. *Using definition of* $KS^p[\mathbb{R}_I^n]$ *we see that* $KS^p[\mathbb{R}_I^n] \subset KS^p[\mathbb{R}_I^{n+1}]$. *Thus we can define*

$$KS^p[\widehat{\mathbb{R}}_I^\infty] = \bigcup_{n=1}^{\infty} KS^p[\mathbb{R}_I^n].$$

Definition 1.9. *If $f \in \mathcal{Z}^p[\mathbb{R}_I^\infty]$, we define the integral of f by*

$$\int_{\mathbb{R}_I^\infty} f(x)d\lambda_\infty(x) = \lim_{n\to\infty} \int_{\mathbb{R}_I^n} f_n(x)d\lambda_\infty(x)$$

where $f_n \in \mathcal{Z}^p[\mathbb{R}_I^n]$ for all n and the family $\{f_n\}$ is a Cauchy sequence.

Theorem 1.10. *If $f \in \mathcal{Z}^p[\mathcal{B}]$ then the integral of f by*

$$\int_{\mathcal{B}} f(y)d\lambda_{\mathcal{B}}(y) = \lim_{n\to\infty} \int_{R_I^n} f_n(x)d\lambda_\infty(x)$$

where $f_n \subset \mathcal{Z}^p[\widehat{\mathcal{B}}]$ for all n and the family $\{f_n\}$ is a cauchy sequence is exists and is unique for every $f \in \mathcal{Z}^p[\mathcal{B}]$

2. Purpose

The Purpose of this paper is to form Zachary space in $\mathbb{R}_I^\infty$, we want to find our space contains BMO$[\mathbb{R}_I^\infty]$ as dense continous embedding. For application of $\mathcal{Z}^p[\mathbb{R}_I^\infty]$, we want to developed $\mathcal{Z}^p[\mathcal{B}]$, where $\mathcal{B}$ is separable Banach space.

3. Space of functions of bounded mean oscillation (BMO$[\mathbb{R}_I^\infty]$)

The space of functions of bounded mean oscillation or BMO naturally arises as the class of functions whose deviation from their mean over cubes is bounded. The space BMO shares similar properties with the space L^∞. In this section we mention few properties of BMO$[\mathbb{R}_I^\infty]$, proof are similar as [GRA].

Definition 3.1. (1) *Let $f \in L_{loc^1}[\mathbb{R}_I^\infty]$ and Q be a cube in $\mathbb{R}_I^\infty$. We define the average of f over Q by avg $f_Q = \frac{1}{\lambda_\infty(Q)} \int_Q f(y)d\lambda_\infty(y)$*

(2) *We define the sharp maximal function $\mathbb{M}^\#(f)(x) = \sup_Q \frac{1}{\lambda_\infty(Q)} \int_Q |f(y) - Avg_Q|d\lambda_\infty(y)$*

If $\mathbb{M}^{\#}(f)(x) \in L^{\infty}[\mathbb{R}_I^{\infty}]$, we say that f is a bounded mean oscillation. More precisely, the space of functions of bounded mean oscillation defined by

$$BMO[R_I^{\infty}] = \{f \in L_l oc[\mathbb{R}_I^{\infty}] \ : \mathbb{M}^{\#}(f) \in L^{\infty}[\mathbb{R}_I^{\infty}]\}$$

and

$$||f||_{BMO} = ||\mathbb{M}^{\#}(f)||_{L^{\infty}}$$

We can also define $BMO[\mathbb{R}_I^{\infty}]$ as:

As $BMO[\mathbb{R}_I^n] \subset BMO[\mathbb{R}_I^{n+1}]$, with $BMO[\widehat{\mathbb{R}}_I^{\infty}] = \bigcup_{k=1}^{\infty} BMO[\mathbb{R}_I^n]$

Theorem 3.2. *$BMO[\mathbb{R}_I^{\infty}]$ is a linear space.*

Proof. Let $f,\ g \in BMO[\mathbb{R}_I^{\infty}]$

Then,

$$\begin{aligned} ||f+g||_{BMO} &= \sup_{Q \subset \mathbb{R}_I^{\infty}} \frac{1}{\lambda_{\infty}(Q)} \int_Q |(f+g)(x) - Avg_Q(f+g)| d\lambda_{\infty} \\ &= \sup_{Q \subset \mathbb{R}_I^{\infty}} \frac{1}{\lambda_{\infty}(Q)} \int_Q |f(x)+g)(x) - Avg_Q f - Avg_Q g| d\lambda_{\infty} \\ &\leq ||f||_{BMO} + ||g||_{BMO} \\ &< \infty \end{aligned}$$

Therefore, $f+g \in BMO[\mathbb{R}_I^{\infty}]$

Again for $\alpha \in \mathbb{C}$,

$$\begin{aligned} ||\alpha f||_{BMO} &= |\alpha| ||f||_{BMO} \\ &< \infty \end{aligned}$$

$$\alpha f \in BMO[\mathbb{R}_I^{\infty}]$$

Hence the result. □

Theorem 3.3. (1) *If $||f||_{BMO} = 0$ then f is a.e. equal to a constant.*

(2) $L^\infty[\mathbb{R}_I^\infty] \subset BMO[\mathbb{R}_I^\infty]$

Proof. For (1) If $||f||_{BMO[\mathbb{R}_I^\infty]} = 0$, then

$$\int_{Q\subset\mathbb{R}_I^\infty} |f(x) - Avg_Q f|d\lambda_\infty = 0$$

Then f has to be a.e. equal to its average A_N over every cube

$$Q = [-N, N]^n \subset \mathbb{R}_I^\infty$$

As $[-N, N]^n \subset [-N-1, N+1]^n$, it follows $A_N = A_{N+1}$

For(2) Let $f \in L^\infty[\mathbb{R}_I^\infty]$

Therefore,

$$Avg_Q|f(x) - Avg_Q f| \leq 2Avg_Q|f|$$

$$\leq 2||f||_{L^\infty}$$

Therefore, $f \in BMO[\mathbb{R}_I^\infty]$ □

Theorem 3.4. (1) *Suppose that there exists an $A > 0$ such that for all cubes $Q \in \mathbb{R}_I^\infty$ there exists a constant C_Q such that*

$$\sup_Q \frac{1}{\lambda_\infty(Q)} \int_Q |f(x) - C_Q|d\lambda_\infty(x) \leq A$$

Then $f \in BMO[\mathbb{R}_I^\infty]$ and $||f||_{BMO[\mathbb{R}_I^\infty]} \leq 2A$

(2) *For all f locally integrable in $\mathbb{R}_I^\infty$, we have*

$$\frac{1}{2}||f||_{BMO[\mathbb{R}_I^\infty]} \leq \sup_Q \frac{1}{\lambda_\infty(Q)} \inf_{C_Q} \int_Q |f(x) - C_Q|d\lambda_\infty(x) \leq ||f||_{BMO[\mathbb{R}_I^\infty]}$$

(3) *If $f \in BMO[\mathbb{R}_I^\infty]$, $h \in \mathbb{R}_I^\infty$ and $\tau^h(f)$ is given by $\tau^h(f)(x) = f(x-h)$ then $\tau^h(f)$ is also in $BMO[\mathbb{R}_I^\infty]$ and $||\tau^h(f)||_{BMO[\mathbb{R}_I^\infty]} = ||f||_{BMO[\mathbb{R}_I^\infty]}$*

Proof. For proof of (1), we use prop7.1.2 p118 of [GRA].

For proof of (2), (3) follows from (1). □

Remark 3.5. *If we consider functions that differ by a constant as equivalent then it is easy to see that $BMO[\mathbb{R}_I^\infty]$ is a Banach space.*

Theorem 3.6. *The duality of $BMO[\mathbb{R}_I^\infty]$ is $H^1[\mathbb{R}_I^\infty]$, where H^1 is Hardy space.*

Proof. As duality of $BMO[\mathbb{R}^n]$ is $H^1[\mathbb{R}^n]$, that is duality of $BMO[\mathbb{R}_I^n]$ is $H^1[\mathbb{R}_I^n]$. we can find duality of $\bigcup_{n=1}^\infty BMO[\mathbb{R}_I^n]$ is $\bigcup_{n=1}^\infty H^1[\mathbb{R}_I^\infty]$

So, duality of $BMO[\mathbb{R}_I^\infty]$ is $H^1[\mathbb{R}_I^\infty]$. □

4. Zachary space $\mathcal{Z}^p[\mathbb{R}_I^\infty]$

We obtain an equivalent definition of BMO$[\mathbb{R}_I^\infty]$ using balls.

Let $\{\mathcal{E}_k(x)\}$ be the family of generating functions for $KS^2[\mathbb{R}_I^\infty]$ and recalling that the characteristic functions for a family of cubes $\{Q_k\}$ centered at each rational point in $\mathbb{R}_I^\infty$. Let $f \in L^1_{loc}[\mathbb{R}_I^\infty]$ and define f_{ak} by

$$f_{ak} = \frac{1}{\lambda_\infty[Q_k]} \int_{Q_k} f(y) d\lambda_\infty(y) = \frac{1}{\lambda_\infty[Q_k]} \int_{Q_k} \mathcal{E}_k(y) f(y) d\lambda_\infty(y).$$

If $p,\ 1 \le p \le \infty$ and $t_k = 2^{-k}$, we define $||f||_{Z^p}$ by

$$||f||_{Z^p[\mathbb{R}_I^\infty]} = \begin{cases} \left(\sum_{k=1}^\infty t_k |\frac{1}{\lambda_\infty[\mathbb{R}_I^\infty]} \int_{\mathbb{R}_I^\infty} [f(y) - f_{ak}] d\lambda_\infty(x)|^p \right)^{\frac{1}{p}}, \text{ for } 1 \le p < \infty; \\ \sup_{k \ge 1} |\frac{1}{\lambda_\infty[\mathbb{R}_I^\infty]} \int_{\mathbb{R}_I^\infty} [f(y) - f_{ak}] d\lambda_\infty(x)|, \text{ for } p = \infty \end{cases}$$

where $x = (x_1, x_2, x_3, ...) \in R_I^\infty$

With $\mathbb{R}_I^\infty = \lim_{n\to\infty} \mathbb{R}_I^n = \bigcup_{n=1}^\infty \mathbb{R}_I^k$

$\mathbb{R}_I^n = \mathbb{R}^n \times I_n$ where $I = [\frac{-1}{2}, \frac{1}{2}]$

The set of functions for which $||f||_{\mathcal{Z}^p} < \infty$ is called the Zachary functions of bounded mean oscillation and order $p,\ 1 \le p \le \infty$.

Theorem 4.1. *Uniqueness and existance of: For $1 \leq p \leq \infty$. a measurable function $f \in \mathcal{Z}^p[\mathbb{R}_I^\infty]$ if there is a Cauchy-sequence $\{f_n\} \subset \mathcal{Z}^p[\widehat{\mathbb{R}}_I^\infty]$ with $f_n \in \mathcal{Z}^p[\mathbb{R}_I^n]$ and $\lim_{n\to\infty} f_n(x) = f(x)$.λ_∞-a.e.*

Proof. Since the family of functions $\{f_n\}$ is Cauchy, it is follows that if the integral exists, it is unique. To prove existence, follow the standard argument and first assume that $f(x) \geq 0$. In this case, the sequence can always be chosen to be increasing, so that the integral exists. The general case now follows by the standard decomposition. □

Theorem 4.2. *If $\mathcal{Z}^p[\mathbb{R}_I^\infty]$ is the class of Zachary functions of bounded mean oscillation and order $p, 1 \leq p \leq \infty$, then $Z^p[\mathbb{R}_I^\infty]$ is linear*

Proof. Let $f, g \in \mathcal{Z}^p[\mathbb{R}_I^\infty]$

Then for $p < \infty$,

$$\begin{aligned} ||f+g||_{\mathcal{Z}^p[\mathbb{R}_I^\infty]} &= \left(\sum_{k=1}^{\infty} t_k |\frac{1}{\lambda_\infty[\mathbb{R}_I^\infty]} \int_{\mathbb{R}_I^\infty} [(f+g)(y) - (f+g)_{ak}] d\lambda_\infty(x)|^p\right)^{\frac{1}{p}} \\ &\leq ||f||_{\mathcal{Z}^p[\mathbb{R}_I^\infty]} + ||g||_{\mathcal{Z}^p[\mathbb{R}_I^\infty} \\ &< \infty \end{aligned}$$

so, $f + g \in \mathcal{Z}^p[\mathbb{R}_I^\infty]$

Again let $\alpha \in \mathbb{C}$ and $f \in \mathcal{Z}^p[\mathbb{R}_I^\infty]$,

$$\begin{aligned} ||\alpha f||_{\mathcal{Z}^p[\mathbb{R}_I^\infty]} &= \left(\sum_{k=1}^{\infty} t_k |\frac{1}{\lambda_\infty[\mathbb{R}_I^\infty]} \int_{\mathbb{R}_I^\infty} [(\alpha f)(y) - (\alpha f)_{ak}] d\lambda_\infty(x)|^p\right)^{\frac{1}{p}} \\ &\leq |\alpha| ||f||_{\mathcal{Z}^p[\mathbb{R}_I^\infty]} \\ &< \infty \end{aligned}$$

Therefore, $\alpha f \in \mathcal{Z}^p[\mathbb{R}_I^\infty]$

For $p = \infty$, it is obvious. Hence the result. □

Theorem 4.3. (1) $||\lambda f||_{\mathcal{Z}^p} \leq |\lambda||f||_{\mathcal{Z}^p}$

(2) $||f + g||_{\mathcal{Z}^p} \leq ||f||_{\mathcal{Z}_p} + ||g||_{\mathcal{Z}_p}$

(3) $||f||_{\mathcal{Z}_p} = 0$ *implies* $f =$*constant(a.s).*

Proof. (1) and (2) are easy, so we ommit the proof part.

For (3), Let $||\mathcal{Z}^p[\mathbb{R}_I^\infty]|| = 0$, Implies

$$|\frac{1}{\lambda_\infty[\mathbb{R}_I^\infty]} \int_{\mathbb{R}_I^\infty} [f(y) - f_{ak}] d\lambda_\infty(x)|^p = 0$$

, This means f has to be a.s. equal to f_{ak} over every balls in $\mathbb{R}_I^\infty$

Remaining fact is from Theirem 3.3(1). □

Remark 4.4. *If we consider functions that differ by a constant as equivalent then it is easy to see that* $\mathcal{Z}^p[\mathbb{R}_I^\infty]$ *is a Banach space and* $\mathcal{Z}^2[\mathbb{R}_I^\infty]$ *is a Hilbert space.*

Theorem 4.5. (1) *The space* $\mathcal{Z}^\infty[\mathbb{R}_I^\infty] \subset \mathcal{Z}^p[\mathbb{R}_I^\infty]$, $1 \leq p \leq \infty$, *as a dense continuous embedding.*

(2) *The space* $BMO[\mathbb{R}_I^\infty] \subset \mathcal{Z}^\infty[\mathbb{R}_I^\infty]$, *as a continuous embedding.*

Proof. For(1), Let $f \in \mathcal{Z}^\infty[\mathbb{R}_I^\infty]$. This implies

$$|\frac{1}{\lambda_\infty[\mathbb{R}_I^\infty]} \int_{\mathbb{R}_I^\infty} [f(y) - f_{ak}] d\lambda_\infty(x)|$$

is uniformly bounded for all k. It follows that $|\frac{1}{\lambda_\infty[\mathbb{R}_I^\infty]} \int_{\mathbb{R}_I^\infty} [f(y) - f_{ak}] d\lambda_\infty(x)|^p$ is uniformly bounded for all p, $1 \leq p < \infty$.

As $\sum_{k=1}^{\infty} t_k = 1$, from the definition of $\mathcal{Z}^p[\mathbb{R}_I^\infty]$, we get

$$\{\sum_{k=1}^{\infty} t_k |\frac{1}{\lambda_\infty[\mathbb{R}_I^\infty]} \int_{\mathbb{R}_I^\infty} [f(y) - f_{ak}] d\lambda_\infty(x)|^p\}^{\frac{1}{p}} < \infty$$

. Therefore $f \in \mathcal{Z}^p[\mathbb{R}_I^\infty]$

For (2), As $\mathrm{BMO}[\mathbb{R}^n] \subset \mathcal{Z}^\infty[\mathbb{R}^n]$ as a dense continous embedding. That is $\mathrm{BMO}[\mathbb{R}_I^n] \subset \mathcal{Z}^\infty[\mathbb{R}_I^n]$ as a dense continous embedding. However $\mathcal{Z}^\infty[\mathbb{R}_I^\infty]$ is the closure of $\bigcup_{n=1}^\infty \mathcal{Z}^\infty[\mathbb{R}_I^n]$. It follows $\mathcal{Z}^\infty[\mathbb{R}_I^\infty]$ contains $\bigcup_{n=1}^\infty BMO[\mathbb{R}_I^n]$ which is dense in $\mathrm{BMO}[\mathbb{R}_I^\infty]$ as its closure. □

5. Zachary space $\mathcal{Z}^p[\mathcal{B}]$, where $\mathcal{B}$ is separable Banach space

Let $\{\mathcal{E}_k(x)\}$ be the family of generating functions for $KS^2[\mathcal{B}]$ and recalling that the characteristic functions for a family of cubes $\{Q_k\}$ centered at each rational point in $\mathcal{B}$. Let $f \in L^1_{loc}[\mathcal{B}]$ and define f_{ak} by

$$f_{ak} = \frac{1}{\lambda_{\mathcal{B}}[Q_k]} \int_{Q_k} f(y) d\lambda_{\mathcal{B}}(y) = \frac{1}{\lambda_{\mathcal{B}}[Q_k]} \int_{Q_k} \mathcal{E}_k(y) f(y) d\lambda_{\mathcal{B}}(y).$$

If $p,\ 1 \le p \le \infty$ and $t_k = 2^{-k}$, we define $||f||_{\mathcal{Z}^p}$ by

$$||f||_{\mathcal{Z}^p[\mathcal{B}]} = \begin{cases} \left(\sum_{k=1}^{\infty} t_k |\frac{1}{\lambda_{\mathcal{B}}[\mathcal{B}]} \int_{\mathcal{B}} [f(y) - f_{ak}] d\lambda_{\mathcal{B}}(x)|^p\right)^{\frac{1}{p}}, \text{ for } 1 \le p < \infty; \\ \sup_{k \ge 1} |\frac{1}{\lambda_{\mathcal{B}}[\mathbb{R}_I^\infty]} \int_{\mathcal{B}} [f(y) - f_{ak}] d\lambda_{\mathcal{B}}(x)|, \text{ for } p = \infty \end{cases}$$

The set of functions for which $||f||_{\mathcal{Z}^p} < \infty$ is called the Zachary functions of bounded mean oscillation and order $p,\ 1 \le p \le \infty$.

We can also construct $\mathcal{Z}^p[\mathcal{B}]$ as:

Let $\mathcal{B}$ be separable Banach space with $S-$ basis.

Let $\mathcal{Z}^p[\widehat{\mathcal{B}}] = \bigcup_{k=1}^\infty \mathcal{Z}^p[\mathcal{B}^k]$

Theorem 5.1. *Uniqueness and existance of: If $f \in \mathcal{Z}^p[\mathcal{B}]$ then the integral of f by*

$$\int_{\mathcal{B}} f(y)d\lambda_{\mathcal{B}}(y) = \lim_{n\to\infty} \int_{R_I^n} f_n(x)d\lambda_\infty(x)$$

where $f_n \subset \mathcal{Z}^p[\widehat{\mathcal{B}}]$ for all n and the family $\{f_n\}$ is a cauchy sequence is exists and is unique for every $f \in \mathcal{Z}^p[\mathcal{B}]$

Proof. Since the family of functions $\{f_n\}$ is Cauchy, it is follows that if the integral exists, it is unique. To prove existence, follow the standard argument and first assume that $f(x) \geq 0$. In this case, the sequence can always be chosen to be increasing, so that the integral exists. The general case now follows by the standard decomposition. □

Theorem 5.2. *If $\mathcal{Z}^p[\mathcal{B}]$ is the class of Zachary functions of bounded mean oscillation and order $p, 1 \leq p \leq \infty$, then $\mathcal{Z}^p[\mathcal{B}]$ is linear*

Proof. Let $f, g \in \mathcal{Z}^p[\mathcal{B}]$

Then for $p < \infty$,

$$\begin{aligned} ||f+g||_{\mathcal{Z}^p[\mathcal{B}]} &= \left(\sum_{k=1}^{\infty} t_k |\frac{1}{\lambda_{\mathcal{B}}[\mathcal{B}]} \int_{\mathcal{B}} [(f+g)(y) - (f+g)_{ak}]d\lambda_{\mathcal{B}}(x)|^p\right)^{\frac{1}{p}} \\ &\leq ||f||_{\mathcal{Z}^p[\mathbb{R}_I^{\mathcal{B}}]} + ||g||_{\mathcal{Z}^p[\mathcal{B}]} \\ &< \infty \end{aligned}$$

so, $f+g \in \mathcal{Z}^p[\mathcal{B}]$

Again let $\alpha \in \mathbb{C}$ and $f \in \mathcal{Z}^p[\mathcal{B}]$,

$$||\alpha f||_{\mathcal{Z}^p[\mathcal{B}]} = \left(\sum_{k=1}^{\infty} t_k |\frac{1}{\lambda_{\mathcal{B}}[\mathcal{B}]} \int_{\mathcal{B}} [(\alpha f)(y) - (\alpha f)_{ak}] d\lambda_{\mathcal{B}}(x)|^p\right)^{\frac{1}{p}}$$

$$\leq |\alpha| ||f||_{\mathcal{Z}^p[\mathcal{B}]}$$

$$< \infty$$

Therefore, $\alpha f \in \mathcal{Z}^p[\mathcal{B}]$

For $p = \infty$, it is obvious. Hence the result. □

Theorem 5.3. (1) $||\lambda f||_{\mathcal{Z}^p} \leq |\lambda| ||f||_{\mathcal{Z}^p}$

(2) $||f+g||_{\mathcal{Z}^p} \leq ||f||_{\mathcal{Z}_p} + ||g||_{\mathcal{Z}_p}$

(3) $||f||_{\mathcal{Z}_p} = 0$ *implies* $f =$ *constant(a.s).*

Proof. (1) and (2) are easy, so we ommit the proof part.

For (3), Let $||\mathcal{Z}^p[\mathcal{B}] = 0$, Implies

$$|\frac{1}{\lambda_{\mathcal{B}}[\mathcal{B}]} \int_{\mathcal{B}} [f(y) - f_{ak}] d\lambda_{\mathcal{B}}(x)|^p = 0$$

, This means f has to be a.s. equal to f_{ak} over every balls in $\mathcal{B}$

Remaining fact is from Theirem 3.3(1). □

Remark 5.4. *If we consider functions that differ by a constant as equivalent then it is easy to see that $\mathcal{Z}^p[\mathcal{B}]$ is a Banach space and $\mathcal{Z}^2[\mathcal{B}]$ is a Hilbert space.*

Theorem 5.5. (1) *The space $\mathcal{Z}^\infty[\mathcal{B}] \subset \mathcal{Z}^p[\mathcal{B}]$, $1 \leq p \leq \infty$, as a dense continuous embedding.*

(2) *The space $BMO[\mathcal{B}] \subset \mathcal{Z}^\infty[\mathcal{B}]$, as a continuous embedding.*

Proof. For(1), Let $f \in \mathcal{Z}^{\infty}[\mathcal{B}]$. This implies

$$|\frac{1}{\lambda_{\mathcal{B}}[\mathcal{B}]}\int_{\mathcal{B}}[f(y)-f_{ak}]d\lambda_{\mathcal{B}}(x)|$$

is uniformly bounded for all k. It follows that $|\frac{1}{\lambda_{\mathcal{B}}[\mathcal{B}]}\int_{\mathcal{B}}[f(y)-f_{ak}]d\lambda_{\mathcal{B}}(x)|^p$ is uniformly bounded for all $p,\ 1 \leq p < \infty$.

As $\sum\limits_{k=1}^{\infty} t_k = 1$, from the definition of $\mathcal{Z}^p[\mathcal{B}]$, we get

$$\{\sum_{k=1}^{\infty} t_k|\frac{1}{\lambda_{\mathcal{B}}[\mathcal{B}]}\int_{\mathcal{B}}[f(y)-f_{ak}]d\lambda_{\mathcal{B}}(x)|^p\}^{\frac{1}{p}} < \infty$$

. Therefore $f \in \mathcal{Z}^p[\mathcal{B}]$

For (2), As $BMO[\mathcal{B}^n] \subset \mathcal{Z}^{\infty}[\mathcal{B}^n]$ as a dense continous embedding. That is $BMO[\mathcal{B}_J^n] \subset \mathcal{Z}^{\infty}[\mathcal{B}_J^n]$ as a dense continous embedding. However $\mathcal{Z}^{\infty}[\mathcal{B}_J^{\infty}]$ is the closure of $\bigcup_{n=1}^{\infty}\mathcal{Z}^{\infty}[\mathcal{B}_J^n]$. It follows $\mathcal{Z}^{\infty}[\mathcal{B}]$ contains $\bigcup_{n=1}^{\infty} BMO[\mathcal{B}_J^n]$ which is dense in $BMO[\mathcal{B}]$ as its closure. □

Theorem 5.6. *$\mathcal{Z}^p[\mathcal{B}]$ and $\mathcal{Z}^p[\mathcal{B}_J]$ are equivalent spaces.*

Proof. If $\mathcal{B}$ is a separable Banach space, T maps $\mathcal{B}$ onto $\mathcal{B}_J \subset R_I^{\infty}$, where T is a isometric isomorphism so that $\mathcal{B}_J$ is a embedding of $\mathcal{B}$ into $\mathbb{R}_I^{\infty}$. This is how we able to define a lebesgue integral on $\mathcal{B}$ using $\mathcal{B}_J$ and T^{-1}. Thus $\mathcal{Z}^p[\mathcal{B}]$ and $\mathcal{Z}^p[\mathcal{B}_J]$ are not different space. □

Theorem 5.7. *$\mathcal{Z}^p[\mathcal{B}] \subset \mathcal{Z}^p[\mathbb{R}_I^{\infty}]$ embedding as closed subspace.*

Proof. As every separable Banach space can be embedded in $\mathbb{R}_I^{\infty}$ as a closed subspace containing $\mathcal{B}_J^{\infty}$,

So, $\mathcal{Z}^p[\mathcal{B}_J^{\infty}] \subset \mathcal{Z}^p[\mathbb{R}_I^{\infty}]$ embedding as a closed subspace. That is $\mathcal{Z}^p[\bigcup_{n=1}^{\infty}\mathcal{B}_J^n] \subset \mathcal{Z}^p[\mathbb{R}_I^{\infty}]$ embedding as a closed subspace.

So, $\mathcal{Z}^p[\mathcal{B}_J^n] \subset \mathcal{Z}^p[\mathbb{R}_I^\infty]$ embedding as a closed subspace. Finally we can conclude that $\mathcal{Z}^p[\mathcal{B}] \subset \mathcal{Z}^p[\mathbb{R}_I^\infty]$ embedding as closed subspace. □

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